AUCTION OF
DIVISIBLE GOODS

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Outline of the seminar

• Relevance of auction theory
• Known results on auction of a single unit good
• Problem with divisible goods
• The research program of auction theory for divisible goods
• Our (me, Pavan and LiCalzi) contribution.
Why auctions are important

• 450% of the world GNP is traded each year by auction.

• Understanding auctions should help us understand the formation of markets by modeling the competition on one side of the market.

• Auctions represent an excellent application of game theory, since in an auction the rules of the game are made explicit.
Most common auctions

Auctions typically take one of four simple forms:

<table>
<thead>
<tr>
<th>Dynamic</th>
<th>Simultaneous Sealed Bid</th>
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<tbody>
<tr>
<td>English (↑ price)</td>
<td>2nd Price</td>
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<tr>
<td>Dutch (↓ price)</td>
<td>≡</td>
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<tr>
<td></td>
<td>1st Price</td>
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Rules of Simple Auctions

- **English**: price increases until only one bidder is left; the remaining bidder gets the good and pays the highest bid.
- **Dutch**: prices decreases until a bidder accepts the price; this bidder gets the good and pays the price at acceptance.
- **Second Price**: each bidder submits a bid in a sealed envelope; the highest bidder gets the good and pays the second highest bid.
- **First Price**: each bidder submits a bid in a sealed envelope; the highest bidder gets the good and pays the amount of his bid.
Models of Private Information

(1) Independent Private Value:
\[ v_i \sim F_i \text{ independently of } v_j \text{ for } j \neq i. \]

(2) Common Value:
\[ e_i = v + \varepsilon_i, \varepsilon_i \sim F_i \text{ w/ mean 0.} \]

(3) Affiliated Value:
\[ v_i(x,s), \text{ my value depends on private information } x = (x_1, ..., x_n) \text{ and state of world } s. \text{ We will not consider this case.} \]
Models of Private Information

• Independent private value model: It makes sense if differences in value arise from heterogeneous preferences over the attributes of the item.

• Common Value: It makes sense if the bidders have homogeneous preferences, so they value the item the same ex post, but have different estimates of this true value.

• Affiliated value model: In this model, each bidder has private information that is positively correlated with the bidder's value of the good.
Benchmark Model

Independent Private Values, Symmetric, Risk Neutral Bidders

- buyer values $v_1, \ldots, v_n \sim F$ on $[0, \infty)$
- seller value $v_0$ (common knowledge)
- order statistics $v_{(1)} \geq v_{(2)} \geq \ldots \geq v_{(n)}$.
- Unique equilibrium in dominant strategies:
  - *English*: bid up to your value or until others stop.
  - *2nd Price*: bid your value.
- The bidder with the highest value wins and pays the second highest value.
Benchmark Model

• The winner gets $v_{(1)} - v_{(2)}$ ex post and expects in the interim state to get:

$$E_{v_{(1)}} \left[ v_{(1)} - v_{(2)} \right] = E \left( \frac{1 - F(v_{(1)})}{f(v_{(1)})} \right)$$

This represents the information rent received by the winner of the auction.

• The seller's expected revenue:

$$E \left( v_{(1)} - \frac{1 - F(v_{(1)})}{f(v_{(1)})} \right)$$
First Price (≡ Dutch)

Symmetric Equilibrium Bidding Strategy

\[ b(v) = v - F(v)^{-(n-1)} \int_0^v F(u)^{n-1} \, du \]

EXAMPLE:

• \( v \sim U \) on \([0,1]\)

• Then \( F(v) = v \), so

\[ b(v) = v - v/n = v(n-1)/n. \]

• The optimal bid converges to the value as \( n \to \infty \), so in the limit the seller is able to extract the full surplus.
Fundamental result for the benchmark auctions: **Revenue Equivalence**

- Seller's revenue:
  
  English = 2nd = 1st = Dutch.
Risk Aversion

The revenue equivalence theorem no longer applies. The seller's revenue from the 1st Price > English.

- English auction: it still is a dominant strategy to bid your value, so the outcome is the same as in the risk neutral case.
- First price auction: a bidder has an incentive to increase his bid from the risk neutral bid, since by increasing the bid, his risk is reduced: he gets a higher probability of winning a smaller prize. This lottery with reduced risk is preferable for the risk averse bidder.

• Competition is greater with the first price auction, as bidders attempt to limit risk by bidding higher.
PROBLEM

Revenue Equivalence (ipv and risk neutrality):
• English = 2nd Price = 1st Price = Dutch.

If the bidders are risk averse then:
• Dutch = 1st Price $\geq$ English = 2nd Price.

How then can we explain the frequent use of the English auction?
• Values are positively correlated (resell of the object is possible).
Common Value

• Each bidder has a private estimate $x_i = v + \varepsilon_i$ of the value $v$, where $\varepsilon_i$ represents the noise in the estimate.

Example:
• A glass jar filled with coins. Each bidder in the room estimates the value of the coins in the jar, but the estimate is imperfect: some overestimate the value of the coins, others underestimate the value.
Common Value

• In a first price auction, you would expect each bidder to make a bid that is an increasing function of the estimate.

• If everyone adopts the same bidding strategy, then the winner of the auction is going to be the bidder that overestimated the value of the coins the most.

• A bidder that does not condition his bid on the assumption that his estimate is the most optimistic among all the bidders will lose money quickly as a result of the winner's curse.
Results: Winner's Curse

I won. Therefore, I overestimated the most. My bid only matters when I win, so I should condition my bid on winning (i.e., that I overestimated the most).

• Winning is bad news about my estimate of value. This is a form of adverse selection that arises in any exchange setting: if you want to trade with me, it must be that no one else offered more, because they did not think that the item is worth what I am willing to pay.
Results:
Value of Private Information

- Rents to the bidders come solely from the privacy of the information and not the quality of the information.
- For example, in an auction with three bidders, if two have the same information and a third has poorer but independent information, then the two with the same information will get a payoff of zero in equilibrium, whereas the one with poorer information gets a positive payoff.
- One implication of this result is that the seller should reveal all his private information.
Results:
Price and Information

- In common value auctions, price tends to aggregate information: as $n \to \infty$, the price converges to the true value if the monotone-likelihood-ratio-property is satisfied.
General Symmetric Model with Affiliated Values

Milgrom and Weber's model:

• $n$ bidders, each with private info $x_i$, which can be thought of as i's estimate of the value.
• Let $x = \{x_1, \ldots, x_n\}$ be the vector of estimates.
• Bidder i's value of the good depends on the state of world $s$ and the private information $x$.
• Let $f(s,x)$ be the joint density, which is symmetric in $x$.
• Bidder i’s value $v_i = u(s,x_i,x_{-i})$ is assumed to be symmetric in $x$, increasing, and continuous.
• Assume values are affiliated: if you have high value, it is more likely that I have high value.
Analysis of Auctions with Affiliated Information

• Throughout the analysis, assume
  1. No collusion.
  2. The choice of auction doesn't reveal info.
  3. The choice of auction doesn't affect who plays.
Analysis of Auctions with Affiliated Information

• For each auction, M&W do the following:
  1. Find the symmetric equilibrium bidding function.
  2. Determine how the seller should use any private information.
  3. Find the order of the simple auctions with respect to the seller's revenue.
Results

• The main result is that in terms of seller revenue:

   English $\geq$ 2nd Price $\geq$ 1st Price = Dutch

• Intuition: The equilibrium bid function depends on everyone's information. The more (affiliated) information you condition on, the higher the bid.
Results

• How does the price depend on the bids in the simple auctions?
  1. 1st Price: only 1st bid
  2. 2nd Price: 1st and 2nd bids
  3. English: all bids

• Hence, the English auction does best because it involves conditioning on the most information. Here it is assumed that the bidders in an English auction observe the point at which each bidder drops out of the auction.
Linkage Principle

A higher price is obtained if the price is linked to more affiliated information.

<table>
<thead>
<tr>
<th>Auction</th>
<th>Condition on</th>
</tr>
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<tbody>
<tr>
<td>1st Price</td>
<td>winner's estimate $X_i \geq Y_1$</td>
</tr>
<tr>
<td>2nd Price</td>
<td>1st &amp; 2nd estimate $X_i \geq Y_1$ &amp; $Y_1 = x_1$</td>
</tr>
<tr>
<td>English</td>
<td>all estimates $X_i \geq Y_1$, $Y_1 = x_1$, $b(Y_j) = p_j$</td>
</tr>
</tbody>
</table>

• Hence, English $\geq$ 2nd Price $\geq$ 1st Price and the seller should always reveal information.
• The more information you condition on the higher is the price.
Auctioning Many Similar Items

CRUCIAL DIFFERENCES WITH PREVIOUS RESULTS
Examples of auctioning similar items

Perfectly divisible goods:
• Treasury bills
• Stock repurchases and IPOs

Peculiar cases:
• Telecommunications spectrum
• Electric power
• Emissions permits
Ways to auction many similar items

- Sealed-bid: bidders submit demand schedules
  - Pay-as-bid auction (traditional Treasury practice)
  - Uniform-price auction (Milton Friedman 1959)
Pay-as-bid Auction: All bids above $P_0$ win and pay bid
Uniform-Price Auction:
All bids above $P_0$ win and pay $P_0$
Payment rule affects behavior

![Diagram showing the relationship between price, quantity, supply, and demand. It illustrates the concepts of Pay-as-bid and Uniform-Price mechanisms. The diagram includes axes labeled as Price, Quantity, Supply, and Demand, with a specific point labeled as $p_0$. The supply and demand curves intersect, indicating the equilibrium price and quantity.]
More ways to auction many similar items

- Ascending-bid: Clock indicates price; bidders submit quantity demanded at each price until no excess demand
  - Standard ascending-bid
  - Ausubel ascending-bid (Ausubel 1997)
More ways to auction many similar items

• Ascending-bid
  – Simultaneous ascending auction (FCC spectrum)

• Sequential
  – Sequence of English auctions (auction house)
  – Sequence of Dutch auctions (fish, flowers)

• Optimal auction
  – Maskin & Riley 1989
General Research Program: How do standard auctions compare?

- Efficiency
  - FCC: those with highest values win

- Revenue maximization
  - Treasury: sell debt at least cost
Efficiency
(not pure common value; capacities differ)

• Uniform-price and standard ascending-bid
  – Inefficient due to demand reduction

• Pay-as-bid
  – Inefficient due to different shading
Inefficiency Theorem

In any equilibrium of uniform-price auction, with positive probability objects are won by bidders other than those with highest values.

- Winning bidder influences price with positive probability
- Creates incentive to shade bid
- Incentive to shade increases with additional units
- Differential shading implies inefficiency
Inefficiency from differential shading

Large Bidder makes room for smaller rival
What about seller revenues?

Price

Supply

Pay-as-bid

Uniform-Price

$Q_i(p)$

$p_0$

$Q_S$

Quantity
OLD IDEAS ON AUCTIONS OF DIVISIBLE GOODS

• Based on the literature on single-unit auctions, a lot of support built up in favor of uniform-price auctions. The main argument was:

• Uniform-price resembles second-price, which in common value settings raises more revenues than first-price mechanism as discriminatory auctions.
PROBLEM

• With divisible goods, participants compete with respect to demand schedules

• Then results based on single-unit auctions may not generalize and actually do not generalize

• In a uniform-price auction, the supply curve faced by a bidder is the residual from the demand of other bidders, i.e. is endogenous

• Bidders in a uniform-price auction can submit steep demand curves, making marginal cost much higher than price for their competitors supporting underpricing equilibria.
Known results with Common Uncertainty

**Proposition.** (Wilson ‘79; Back & Zender ‘93)

- *Wide range of prices can be supported as equilibrium in uniform-price auction, even if supply is stochastic; highest yields EV.*

**Proposition.** (Wang & Zender ‘96)

- *Many equilibria in pay-as-bid auction, even if supply is stochastic; highest yields EV*
- *Indeterminacy avoided if set reserve price (even 0)*
A figure showing underpricing equilibria
But uniform price has advantages

• Participation
  – Encourages participation by small bidders (since quantity is shifted toward them)
  – May stimulate competition

• Post-bid competition
  – More diverse set of winners may stimulate competition in post-auction market
Our contribution: within uniform-price auctions is it possible to enhance seller’s revenue?

• The underpricing equilibria of uniform-price auctions exists because bidders use strategically their demand schedules:

• Bidders submit high inframarginal bids to prevent competition on prices (“collusive” behaviour)

• This has no cost because of the payment rules of uniform-price auctions
The seller as a strategic player

• A common assumptions for previous results is that supply is fixed in advance
• This implies a strategic asymmetry between bidders and sellers: the former can optimally use their demand schedule to inhibit price competition, but the latter cannot choose her supply to enhance it
• In general the amount of divisible good on offer may change endogenously with its uniform price
Elastic Supply

- Some underpricing equilibria are no longer self-enforcing if the seller announces and increasing supply schedule, since
  - If steepness of bidders’ demand curves has a price effect which increases the marginal cost of a higher bid
  - An increasing supply function induces a quantity effect that raises bidders’ marginal revenues
- The general idea is to allow the seller to use the supply schedule as a strategic instrument to reward aggressive bidding and enhance her expected revenue
Related independent papers

• Lengwiler 1998
• Back and Zender 1999
• McAdams 2000

• These papers do not specifically address the optimality of an elastic supply for uniform-price auction, the main contribution of this paper
THE MODEL

• Two stage game:
  – The seller (female) announces the supply schedule
  – Bidders submit their demand functions

• n risk-neutral symmetric bidders (males)

• Each bidder has pure common value v for security and can purchase any quantity (flat demand curve w/o capacity)

• To simplify: Common uncertainty
  – Bidders have no private information

• The seller knows that v is distributed on [vL, vH] with c.d.f. F(v): asymmetric information between bidders and seller.
Strategic variables of players

- Bidder i submits a weakly decreasing left-continuous demand schedule
  \[ d_i(p), \quad D(p) = \sum_{i=1}^{n} d_i(p) \]

- The seller sets a reserve price \( p_L \geq 0 \) and an increasing and left-continuous supply schedule \( S(p) \)

- There are no competitive bidders and hence no noise

- The secondary market plays no explicit role
Stop-out price

- The stop-out price $P$ is the market clearing price, but to deal with flats or failure of intersections, the definition is slightly complex:

If $\exists p \geq p_L : S(P) \geq D(P)$, then $P = \max\{p \geq p_L \mid D(p) \geq S(p)\}$

otherwise $P = \sup\{\arg\max_{p \geq p_L} D(p)\}$
Rationing

• With flats in the individual bidding and in the aggregate demands, the aggregate demand may exceed supply at the stop out price, so we assume pro rata rationing: the quantity assigned to bidder $i$ is:

$$\hat{d}_i(P) = d_i(P) - \lambda \Delta d_i(P),$$

where

$$\Delta d_i(P) = d_i(P) - \lim_{p \to p^+} d_i(p)$$

$$\lambda = \max \left\{ \frac{D(P) - S(P)}{\Delta D(P)} , 0 \right\}$$

$$\Delta D(P) = \sum_{i=1}^{n} \Delta d_i(P)$$
The players payoff functions

- Bidder $i$ payoff function is:
  \[ \pi_i = (v - P) \hat{d}_i(P) \]

- The seller payoff function is:
  \[ \pi^S = PQ - C(Q) \]

- When supply is constrained to be fix, then maximizing $\pi^S$ is consistent with revenue maximization;

- This formulation allows to take into account the costs of issuing different quantities, e.g.
  - Cost of issuing debt for Treasury auctions
  - Costs of diluting control for IPO
PROPOSITION 1: Existence of underpricing equilibria

Assume \( v \geq p_L \) and a fixed supply \( S(p)=Q \). For any price \( p^* \) in \([p_L, v]\), there exists a symmetric Nash equilibrium in pure strategies such that the stop-out price is \( P=p^* \).

The equilibrium demand schedule is

\[
d_i^*(p) = \begin{cases} 
0 & \text{if } p > v \\
y(p) & \text{if } p^* < p \leq v \\
\frac{Q}{n-1} & \text{if } p_L \leq p \leq p^*
\end{cases}
\]

Where \( y(p) \) is any non increasing, a convex function on \((p^*,v]\) left-continuous at \( p=v \) such that

\[
\lim_{p \to p^*+} y(p) = \frac{Q}{n} \quad \text{and} \quad \lim_{p \to p^*+} y'(p) > \frac{-Q}{n(n-1)(v-p^*)}
\]
Comments to proposition 1

- Specifies a wider class of equilibrium strategies wrt th.1 in B&Z 1993
- Other choices for the equilibrium strategies are possible, but these are effective for next results because they minimize the competitors’ residual supply
- Note that for $p_L \leq v_L$, the bidders earn positive profits with certainty
- Proposition 1 allow for a continuum of stop-out price, but the most prominent is $P=p_L$, the one selected by the coalition-proofness refinement.
PROPOSITION 2: Elasticity of supply and underpricing equilibria

Assume \( v \geq p_L \) and an increasing, absolutely continuous supply \( S(p) \). Given \( p^* \) in \([p_L, v]\), let

\[
\delta(p) = -py'(p) / y(p), \quad \delta(p^*) = \lim_{p \to p^+} \delta(p)
\]

\[
\alpha(p^*, v, n) = \frac{1}{n} \left[ \frac{p^*}{v - p^*} - (n - 1)\delta(p^*) \right] > 0
\]

\[
\sigma(p) = -pS'(p) / S(p), \quad \sigma(p^*) = \lim_{p \to p^+} \sigma(p)
\]

If \( \sigma(p^*) > \alpha(p^*, v, n) \), then coordination on \( p^* \) by submitting the profile of demand schedule \( \{d^*(p)\} \) is no longer an equilibrium for the \( n \) bidders.
Comments to proposition 2

• The scope of prop. 2 is not to suggest that the seller should compute $\alpha$ to undermine an underpricing equilibrium, since she does not know $v$.

• Prop. 2 simply shows that bidders’ coordination on $p^*$ using the demands of prop. 1 is not enforceable when

$$\frac{v - p^*}{p^*} > \left[ n \sigma (p^*) + (n - 1) \delta (p^*) \right]^{-1}$$

• Note that $\alpha$ is decreasing in $n$

• Proposition 2 suggests that the seller may have an incentive to strategically use an elastic supply schedule.
PROPOSITION 3: Equilibria under increasing supply

Assume \( v \geq p_L \) and an increasing continuous supply \( S(p) \). Let \( T \) be the set of all stop-out prices that can be supported by some symmetric Nash equilibrium where players submit decreasing demand schedules. Consider the set \( T' \) of all stop-out prices \( p^* \) than are supported by the following profile of symmetric demand schedules

\[
d_i^*(p) = \begin{cases} 
0 & \text{if } p > v \\
S(p^*/n) & \text{if } p^* < p \leq v \\
\frac{S(p^*)}{n-1} & \text{if } p_L \leq p \leq p^* 
\end{cases}
\]

Then \( T = T' \).
Comments to proposition 3

• This proposition characterizes the set of all prices that can be supported as a symmetric equilibrium in a uniform price auction when the supply is increasing;

• Proposition 3 is a simple but powerful result: although there can be other profiles of demand schedules that sustain the same outcome, there is no loss of generality in limiting our attention to the inelastic schedules of this proposition;
• The intuition behind proposition 3 is the following: an increasing supply induce to bid more aggressively, then to sustain a low price bidders need to compensate the positive quantity effect by reducing the residual supply for competitors and this is most effectively done by submitting perfectly inelastic demand schedules.
THE SELLER AS A STRATEGIC PLAYER

Assumptions for tractability

• \( C(Q) = \alpha + \beta(Q - \bar{Q}) \), with \( \alpha \geq 0, \beta > 0, \bar{Q} \) target quantity;

• The seller can choose any reserve price \( p_L \geq 0 \) and any increasing linear supply function

\[
S(p) = \begin{cases} 
  r + s(p - p_L) & \text{if } p \geq p_L \\
  0 & \text{otherwise.}
\end{cases}
\]

• The triple \( \{p_L, r, s\} \) of positive real numbers is the set of strategic variable that defines a linear supply mechanism for a uniform price auction.
The two stage game

- **First stage:** the seller chooses a (linear) supply mechanism to maximize her payoff

- **Second stage:** the bidders observe the supply mechanism and choose a demand schedule to maximize their payoffs

- Solve the game backward
PROPOSITION 4: the set of equilibrium stop-out prices

In the two stage game, for a given supply mechanism \{p_L, r, s\} the set of possible equilibrium stop-out prices is the interval \([pc, \nu]\), where

\[
p_c = \max \left\{ p_L, \frac{nv + p_L}{n + 1} - \frac{r}{(n + 1)s} \right\}
\]
Comments to proposition 4

• The minimum price $p_c$ is increasing in the number of bidders, in the limit $p_c$ is equal to $v$: with an increasing supply, if the number of bidders were infinite, the seller could extract all the surplus from the bidders. This result contrasts with the case of fixed supply, where the number of bidders does not affect the equilibrium set.

• The minimum collusive price $p_c$ is the unique coalition-proof equilibrium, and we will use this price to solve the first stage, looking for the seller optimal supply mechanism.
PROPOSITION 5: Existence of optimal supply mechanism

Suppose that the seller believes \( v \) to be uniformly distributed in \([0, 1]\).

The optimal linear supply mechanism exists and it is strictly increasing for \( p \geq p_L \), with \( s^* \) strictly positive and finite.
Comments to proposition 5

- Increasing the elasticity of the supply schedule enhances price competition among bidders and leads to a higher equilibrium price. On the other hand, making the supply schedule more elastic contrasts with the objective of maintaining control on the total quantity auctioned. There is thus an obvious trade-off between price competition and quantity control: proposition 5 states that it is not struck at either extreme;

- If the simple assumption of a linear supply mechanism suffices to rule out the optimality of a constant supply, this true \textit{a fortiori} for more general supply mechanism.
PROPOSITION 6: characterization of optimal supply mechanism

Suppose that the seller believes $v$ to be uniformly distributed in $[0, 1]$. Assume $n=2$ and a cost function $C(Q) = a + bQ^2$, with $a \geq 0, b > 0, \bar{Q} = 0$. The optimal linear supply mechanism has

$$p_L = \frac{2}{5}, \quad r = 0 \quad \text{and} \quad s = \frac{5}{4b}$$

and the supply schedule is

$$S(p) = \max\left\{ \frac{5}{4b} p - \frac{1}{2b}, 0 \right\}.$$
Comments to proposition 6

• The optimal reserve price is set below the bidders’ expected value (E(v)=1/2): when the supply is not constrained to be fixed, the seller gets higher profits by reducing the risk of not selling the good while inhibiting bidders’ coordination on low prices;

• The slope of the supply schedule decreases as b increases: a fixed supply is optimal when b goes to zero and a perfectly elastic supply is optimal as b goes to infinity. These are obviously extreme non realistic cases.
CONCLUDING REMARKS

This paper suggests

– The practice to combine uniform-price auction with fixed supply with a reserve price below the market value can be suboptimal for the Treasury. The adoption of an elastic supply with an appropriate reserve price may raise higher expected revenues;

– The use of an elastic supply would enable the seller to exploit a positive correlation between the supply of shares and the stop-out prices: successful auctions would be associated to higher issuances, unsuccessful auctions would turn out in lower issuances.
Another prominent example of using uniform-price auctions concern internet-based and Israel IPOs. The empirical evidence e.g. in Kandel et alii 1998 have found significant underpricing with demand schedules that have a flat around the IPO price and are very similar to those described in Proposition 1. Our analysis shows that a company which goes public has a simple way to reduce the possibility of large underpricing and raise more money: instead of announcing a fixed supply, the company should make the supply of shares a function of the stop-out price.